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The motion of a railway wheelset☆

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ABSTRACT

The rolling of a railway wheelset along rails without slipping is investigated taking the creep hypothesis into account. The wheelset is represented by two cones that have a common base, and the rails are represented by two circular cylinders with parallel axes. The kinematic characteristics of the unperturbed rolling motion of the wheelset, which occurs when the centre of mass moves along a straight line, and of the perturbed motion, which occurs when the centre of mass of the wheelset describes a sinusoidal trajectory, are determined. The constraint reactions are found for the motions investigated up to small second-order values of the perturbed variables. When the elastic properties of the material in the contact area are taken into account, the creep hypothesis is used, averaging over the fast variables is employed, and the value of the critical speed, above which the rectilinear rolling of the wheelset becomes unstable, is found using averaged equations. In the latter case a periodic mode with two time intervals when the wheel flanges come into contact with the rails is investigated. The reaction force, the work of the dry friction force, and the moment of the active forces needed to maintain the periodic mode are found at the flange/rail contact point within the dry friction model. The boundaries of the stability regions, the parameters of the periodic mode and the moment of the resistance forces as functions of the problem parameters are determined from the formulae obtained by analytical methods.

A railway wheelset, which consists of two wheels joined by an axle, is an important component of carriages and locomotives for constructing mechanical models. In the case of a symmetrical placement of a wheelset on rails, its centre of mass, which is at the middle of the wheelset axle, moves along a straight line as the wheels roll. When the symmetry is violated, the trajectory of the centre of mass, projected onto a horizontal plane, takes a sinusoidal form. The motion of a railway wheelset is similar to the shimmy of the wheels of an aircraft rolling on a runway.[1] This motion, which corresponds to rolling without slipping, was analysed in 1883 by Klingel[2] and was subsequently investigated.[3,4] Numerous recent studies on this subject are based on numerical simulation using Kalker's models to describe the wheel/rail interaction.[5] It should be noted that these models are based on the creep hypothesis.[3,4] Rocard used the creep hypothesis to describe the dynamics of wheels with pneumatic tyres.[1,5]

1. Statement of the problem. Holonomic and differential constraints

We will consider the model of a wheelset of a railway carriage. We will specify the parts of the wheel surfaces that come in contact with the rail by the conical surfaces Σ_k, and we will represent the surfaces of the rails in the form of two circular cylinders Π_k ($k = 1, 2$). Let $OX_1X_2X_3$ be a fixed system of coordinates, relative to which the surfaces of the rails are specified by the equations (Fig. 1)

$$\Pi_k : [X_2 + (-1)^k(l + b\sin\beta]^2 + (X_3 + r + b\cos\beta)^2 = b^2, \quad k = 1, 2$$

where $k = 1$ corresponds to the right-hand rail, $k = 2$ corresponds to the left-hand rail, r and b are the radii of a wheel and a rail, $2l$ is the distance between the wheel/rail contact points when the wheelset axle is parallel to the OX_2 axis, and β is the coning angle of the wheels.

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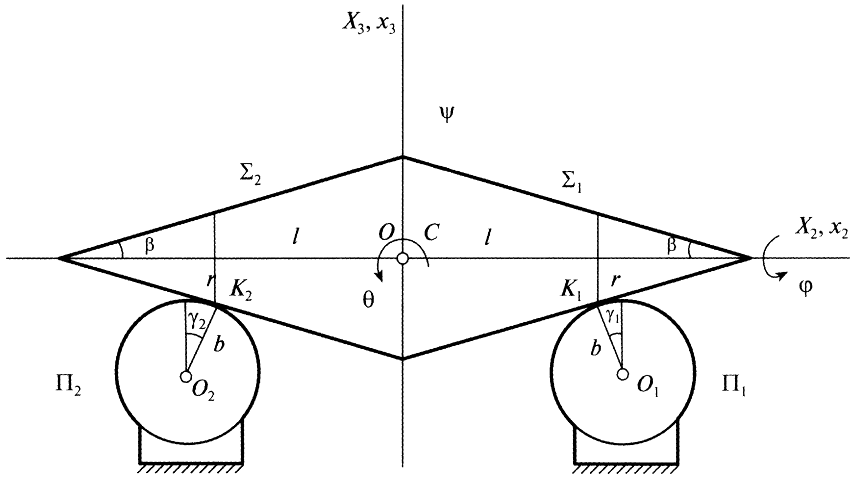

Fig. 1.

We attach the fixed system of coordinates $Cx_1x_2x_3$ to the wheelset and represent the surfaces of the wheels relative to it by the equations

$$\Sigma_k : \begin{cases} x_{1k} = -(r + u_k \sin\beta)\sin\varphi_k \\ x_{2k} = (-1)^{k+1}(l - u_k \cos\beta)\ ; \quad k = 1, 2 \\ x_{3k} = -(r + u_k \sin\beta)\cos\varphi_k \end{cases}$$

where u_k and φ_k are parameters. We will specify the position of the fixed system of coordinates $Cx_1x_2x_3$, which is attached to the wheelset, relative to the fixed system of coordinates $OX_1X_2X_3$ by the displacement vector of its origin $\mathbf{R}_C = \sum_{i=1}^{3} X_i\xi_i$, where ξ_i is the unit vector OX_i, and by the rotation matrix $\Gamma_1(\theta)\ \Gamma_3(\psi)\ \Gamma_2(\varphi)$, where

$$\Gamma_1(\theta) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{Vmatrix}, \Gamma_3(\psi) = \begin{Vmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \Gamma_2(\phi) = \begin{Vmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{Vmatrix}$$

Fig. 1 shows a cross section of the rails and the wheelset, in which the systems of coordinates $OX_1X_2X_3$ and $Cx_1x_2x_3$ are identical.

Let K_1 be the point of contact of the right-hand wheel with the right-hand rail. On the rail this point corresponds to the cylindrical coordinates (Z_1, γ_1), and its radius vector in the fixed system of coordinates is represented in the form

$$\mathbf{R}_{1r} = Z_1\xi_1 + [l + b(\sin\beta - \sin\gamma_1)]\xi_2 + [b(\cos\gamma_1 - \cos\beta) - r]\xi_3 \tag{1.1}$$

The polar angle γ_1 is close to the angle β and is measured from the O_1X_3 axis, which passes through the point O_1, i.e., the centre of the circle that forms the cylindrical surface of the right-hand rail, and is parallel to the OX_3 axis. On the other hand, the contact point K_1 belongs to the conical surface of the right-hand wheel, and its radius vector in the fixed system of coordinates is represented in the form

$$\mathbf{R}_{1w} = \sum_{i=1}^{3} X_i\xi_i + \Gamma_1(\theta)\Gamma_3(\psi)\Gamma_2(\varphi + \varphi_1)[(l - u_1\cos\beta)\mathbf{e}_2 - (r + u_1\sin\beta)\mathbf{e}_3] \tag{1.2}$$

Here $\mathbf{e}_i$ is a column vector with unity in the i-th place and zeros in the remaining places. We obtain similar relations for the point K_2, i.e., the point of contact of the left-hand wheel with the rail

$$\mathbf{R}_{2r} = Z_2\xi_1 - [l - b(\sin\gamma_2 + \sin\beta)]\xi_2 - [b(\cos\beta - \cos\gamma_2) + r]\xi_3 \tag{1.3}$$

$$\mathbf{R}_{2w} = \sum_{i=1}^{3} X_i\xi_i - \Gamma_1(\theta)\Gamma_3(\psi)\Gamma_2(\varphi + \varphi_2)[(l - u_2\cos\beta)\mathbf{e}_2 + (r + u_2\sin\beta)\mathbf{e}_3] \tag{1.4}$$

The angle γ_2 is negative, is close to the angle $-\beta$ and is measured from the O_2X_3 axis, which passes through the point O_2, i.e., the centre of the circle that forms the cylindrical surface of the left-hand rail and is parallel to the OX_3 axis. When the wheelset axle is parallel to the OX_2 axis (the symmetrical position of the wheelset on the rails), we have $u_1 = u_2 = 0$. Equating the right-hand sides of (1.1) and (1.2) and of (1.3) and (1.4), we obtain six scalar equations in generalized coordinates, which specify the positions of the wheel/rail contact points.

The second condition that holds at the contact points is that the surfaces of the wheels and the rails at these points have common tangential planes and, consequently, the corresponding normals to these surfaces are collinear:

$$\frac{\partial\mathbf{R}_{kw}}{\partial u_k} \times \frac{\partial\mathbf{R}_{kw}}{\partial\varphi_k} = \lambda_k\left(\frac{\partial\mathbf{R}_{kr}}{\partial Z_k} \times \frac{\partial\mathbf{R}_{kr}}{\partial\gamma_k}\right), \quad k = 1, 2 \tag{1.5}$$

Here the λ_k are coefficients of proportionality. We introduce the notation $\varphi+\varphi_k=\hat{\varphi}_k$ and note that when the wheelset rolls along the rails, the angles θ, ψ, $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are small. A similar statement holds for the dimensionless components of the displacement vector of the centre of mass of the wheelset X_2/l and X_3/l and for the quantities u_1/r and u_2/r, which specify the displacements of the contact points along the generatrices of the conical surfaces of the wheels. Utilizing this situation and retaining the terms that are of the first-order in these variables, we represent the right- and left-hand sides of relations (1.5) in the form

$$\frac{\partial \mathbf{R}_{1w}}{\partial u_1}\times\frac{\partial \mathbf{R}_{1w}}{\partial \varphi_1}=\Gamma_1(\theta)\Gamma_3(\psi)\Gamma_2(\hat{\varphi}_1)[(\cos\beta\mathbf{e}_2+\sin\beta\mathbf{e}_3)\times[\mathbf{e}_2\times(r+u_1\sin\beta)\mathbf{e}_3]]\approx$$
$$\approx(r+u_1\sin\beta)\begin{Vmatrix}1 & -\psi & \hat{\varphi}_1\\ \psi & 1 & -\theta\\ -\hat{\varphi}_1 & \theta & 1\end{Vmatrix}[(\cos\beta\mathbf{e}_2+\sin\beta\mathbf{e}_3)\times\mathbf{e}_1]=$$
$$=(r+u_1\sin\beta)[-(\psi\sin\beta+\hat{\varphi}_1\cos\beta)\xi_1+(\sin\beta+\theta\cos\beta)\xi_2+(-\cos\beta+\theta\sin\beta)\xi_3]$$

$$\frac{\partial \mathbf{R}_{1r}}{\partial Z_1}\times\frac{\partial \mathbf{R}_{1r}}{\partial \gamma_1}=-\xi_1\times b[\cos\gamma_1\xi_2+\sin\gamma_1\xi_3]=b(\sin\gamma_1\xi_2-\cos\gamma_1\xi_3) \tag{1.6}$$

From conditions (1.5), taking into account relations (1.6), for $k=1$ (contact of the right-hand wheel) we obtain the equalities

$$\hat{\varphi}_1\approx-\psi\,\mathrm{tg}\beta,\quad \gamma_1\approx\beta+\theta \tag{1.7}$$

For the left-hand wheel ($k=2$), similar mathematical operations lead to the equalities

$$\hat{\varphi}_2\approx\psi\,\mathrm{tg}\beta,\quad \gamma_2\approx-\beta+\theta \tag{1.8}$$

We will write the relations $\mathbf{R}_{kw}=\mathbf{R}_{kr}$ ($k=1, 2$) in the form of a system of six equations, retaining the small first-order terms, taking equalities (1.7) and (1.8) into account,

$$X_1+(-1)^k\psi l=Z_k+r\hat{\varphi}_k$$
$$X_2+\theta(r+b\cos\beta)=-(-1)^k u_k\cos\beta$$
$$X_3-(-1)^k\theta(l+b\sin\beta)=u_k\sin\beta;\quad k=1,2 \tag{1.9}$$

The equality $u_1+u_2=0$ follows from the second equality in (1.9), and the equalities $2X_3=(u_1+u_2)\sin\beta=0$ and $u_1=-u_2=(b+l/\sin\beta)\theta$ follow from the third equality. Then we find

$$X_2=(l\,\mathrm{ctg}\beta-r)\theta,\quad Z_1=X_1-(l-r\,\mathrm{tg}\beta)\psi,\quad Z_2=X_1+(l-r\,\mathrm{tg}\beta)\psi \tag{1.10}$$

The relations obtained show that all the variables can be expressed in terms of the three quantities X_1, ψ and θ. Note that the motion of the wheelset along the rails without slipping is instantaneous rotational motion of a rigid body, since the axis of instantaneous rotation passes through the contact points K_1 and K_2. The latter means that the angular velocity vector of the wheelset is collinear with the vector

$$\mathbf{R}_{1r}-\mathbf{R}_{2r}=(Z_1-Z_2)\xi_1+[b(\sin\gamma_1-\sin\gamma_2+2\sin\beta)+2l]\xi_2+b(\cos\gamma_1-\cos\gamma_2)\xi_3$$

The angular velocity of the wheelset in the fixed system of coordinates is as follows:

$$\mathbf{\Omega}=\dot{\theta}\xi_1+\dot{\psi}\Gamma_1(\theta)\mathbf{e}_3+\dot{\varphi}\Gamma_1(\theta)\Gamma_3(\psi)\mathbf{e}_2\approx(\dot{\theta}-\dot{\varphi}\psi)\xi_1+\dot{\varphi}\xi_2+(\dot{\psi}+\dot{\varphi}\theta)\xi_3$$

The conditions for collinearity of the vectors found have the form

$$\frac{\dot{\theta}-\dot{\varphi}\psi}{-(l-r\,\mathrm{tg}\beta)\psi}=\frac{\dot{\varphi}}{l}=\frac{\dot{\psi}+\dot{\varphi}\theta}{-b\theta\sin\beta} \tag{1.11}$$

Equalities (1.11) lead to the equations

$$\dot{\theta}-k_1\dot{\varphi}\psi=0,\quad \dot{\psi}+k_2\dot{\varphi}\theta=0,\quad k_1=rl^{-1}\mathrm{tg}\beta>0,\quad k_2=(l+b\sin\beta)l^{-1}>0 \tag{1.12}$$

Assuming that $\dot{\varphi}>0$ and denoting the derivative with respect to the angle φ by a prime, we represent the system of equations (1.12) in the complex form

$$W'+i\sqrt{k_1k_2}W=0,\quad W=\theta+i\psi\sqrt{k_1k_2^{-1}}\Rightarrow W=q\exp[-i(\vartheta+\alpha)],\ \vartheta=\varphi\sqrt{k_1k_2} \tag{1.13}$$

where q and α are arbitrary real constants. Equating the real and imaginary parts in Eq. (1.13), we obtain

$$\theta=q\cos(\vartheta+\alpha),\quad \psi=-q\sqrt{k_2k_1^{-1}}\sin(\vartheta+\alpha) \tag{1.14}$$

One more differential constraint should be noted. The velocities of points of the wheelset that lie on the axis of instantaneous rotation are equal to zero when they are projected onto that axis (contact of the wheels without slipping). The corresponding condition has the form

$$\mathbf{V}_C\cdot\overline{K_1K_2}+[\mathbf{\Omega}\times\overline{CK_1}]\cdot\overline{K_1K_2}=0 \tag{1.15}$$

The mechanical system under consideration is a holonomic system with one degree of freedom, since the number of parameters that define the position of the system is equal to 14 (X_1, X_2, X_3, θ, ψ, φ, u_1, φ_1, u_2, φ_2, Z_1, γ_1, Z_2, γ_2), and the number of relations between these parameters is equal to 13 (four conditions of the form (1.5), six conditions of the form (1.9), two conditions of the form (1.12), and one condition of the form (1.15)). The linearized differential constraints (1.11) and (1.15) are integrated explicitly, and the system can be treated as a system with holonomic constraints that depend on the initial conditions of motion. This dependence manifests itself in formulae (1.14) as a dependence on the arbitrary constants q and α. The angle φ, i.e., the rotation angle of the wheelset about the Cx_2 axis, can be taken as a generalized coordinate. The velocity of the centre of mass of the wheelset equals

$$\mathbf{V}_C = \mathbf{\Omega}\times(\mathbf{R}_C - \mathbf{R}_{1w})$$

$$\mathbf{R}_C - \mathbf{R}_{1w} = -\Gamma_1(\theta)\Gamma_3(\psi)\Gamma_2(\hat{\varphi}_1)[(l - u_1\cos\beta)\mathbf{e}_2 - (r + u_1\sin\beta)\mathbf{e}_3] \tag{1.16}$$

We differentiate equality (1.16) with respect to time, taking into account the time dependence of $\hat{\varphi}_1$ and u_1. As a result, taking into account only the first-order terms, we obtain

$$\ddot{X}_1 = r\ddot{\varphi} + l\ddot{\psi} + \dot{\varphi}\dot{\theta}(l + b\sin\beta), \quad \ddot{X}_2 = -r(\ddot{\theta} - \dot{\varphi}\dot{\psi}), \quad \ddot{X}_3 = -l\ddot{\theta} + r\dot{\varphi}\dot{\psi}\mathrm{tg}\beta \tag{1.17}$$

Next, we use Eq. (1.12) to find $\ddot{X}_1 = r\ddot{\varphi}$ and $\ddot{X}_3$. Taking into account the first equalities in (1.10) and (1.14), we can state that the projections of the wheel/rail contact points onto the OX_1X_2 plane describe sinusoids, whose amplitudes and phases depend on the initial conditions of motion, apart from small higher-order quantities. The period of the sinusoidal wave is $L = 2\pi r/\sqrt{k_1k_2}$. The variation of the angle φ with time is determined from Lagrange's equation of the second kind taking into account the external forces and the moments acting on the wheelset, or from the law of conservation of energy if the external forces are conservative, since the constraints imposed are ideal.

2. Determination of the constraint reactions

To construct a model of the wheel/rail interaction, we need to know the reactions $\mathbf{N}_1$ and $\mathbf{N}_2$ at the contact points K_1 and K_2. We return to the original problem, eliminate the constraints at the contact points and write theorems describing the variation of the momentum and angular momentum about the centre of mass of the wheelset

$$\begin{aligned}
&m\ddot{X}_i = (\mathbf{N}_1 + \mathbf{N}_2)\xi_i - mg\delta_{3i}\xi_i, \quad i = 1, 2, 3\\
&J(\dot{\boldsymbol{\omega}}_1 + \ddot{\varphi}\mathbf{e}_2) + \boldsymbol{\omega}_1\times J(\boldsymbol{\omega}_1 + \dot{\varphi}\mathbf{e}_2) =\\
&= \Gamma_3(-\psi)\Gamma_1(-\theta)[\overrightarrow{CK_1}\times\mathbf{N}_1 + \overrightarrow{CK_2}\times\mathbf{N}_2] + M_2\mathbf{e}_2\\
&J = \mathrm{diag}\{A, B, A\}, \quad \boldsymbol{\omega}_1 = \dot{\theta}\Gamma_3(-\psi)\mathbf{e}_1 + \dot{\psi}\mathbf{e}_3
\end{aligned} \tag{2.1}$$

where δ_{3i} is the Kronecker delta. Using Eq. (2.1), we find the projections of the constraint reactions onto the axis of the fixed system of coordinates, taking into account only the first-order small quantities θ, $\dot{\theta}$, ψ and $\dot{\psi}$ and assuming that $\dot{\varphi} = \omega = \mathrm{const}$. Constancy of the angular velocity $\dot{\varphi}$ can be achieved by appropriately selecting the moment of the external forces M_2. We represent Eq. (1.14) in the form

$$\theta = q\cos(\vartheta + \alpha), \quad \psi = -\sqrt{\frac{k_2}{k_1}}q\sin(\vartheta + \alpha), \quad \vartheta = \omega t\sqrt{k_1k_2} \tag{2.2}$$

We find the constraint reactions in the form $\mathbf{N}_k = \mathbf{N}_{k0} + \Delta\mathbf{N}_k$ ($k = 1, 2$), where the reactions $\mathbf{N}_{k0}$ correspond to unperturbed rectilinear rolling of the wheelset and are determined from the equations of the zero approximation of Eq. (2.1)

$$\mathbf{N}_{10} + \mathbf{N}_{20} = mg\xi_3, \quad \xi_2\times(\mathbf{N}_{10} - \mathbf{N}_{20}) = 0$$

Assuming that the reactions are orthogonal to the wheel/rail contacting surfaces in the case of rectilinear rolling of the wheelset, we obtain

$$\mathbf{N}_{10}\xi_1 = \mathbf{N}_{20}\xi_1 = 0, \quad \mathbf{N}_{10}\xi_2 = -\mathbf{N}_{20}\xi_2 = -mg\mathrm{tg}\beta/2, \quad \mathbf{N}_{10}\xi_3 = \mathbf{N}_{20}\xi_3 = mg/2$$

Perturbations of the reactions satisfy the equations of the first approximation (the first-order terms) of system of equations (2.1)

$$\begin{aligned}
&m\ddot{X}_i = (\Delta\mathbf{N}_1 + \Delta\mathbf{N}_2)\xi_i, \quad i = 1, 2, 3\\
&A\ddot{\theta} - B\omega\dot{\psi} = mg\left(r - \frac{b}{\cos\beta} - l\mathrm{ctg}\beta\right)\theta + l(\Delta\mathbf{N}_1 - \Delta\mathbf{N}_2)\xi_3 + r(\Delta\mathbf{N}_1 + \Delta\mathbf{N}_2)\xi_2\\
&0 = M_2 - r(\Delta\mathbf{N}_1 + \Delta\mathbf{N}_2)\xi_1\\
&A\ddot{\psi} + B\omega\dot{\theta} = mgl\mathrm{tg}\beta(1 - k_1)\psi - l(\Delta\mathbf{N}_1 - \Delta\mathbf{N}_2)\xi_1
\end{aligned} \tag{2.3}$$

When relations (2.2) are taken into account, the projections of the acceleration of the centre of mass of the wheelset (1.17) are as follows:

$$\ddot{X}_1 = 0, \quad \ddot{X}_2 = -r\omega^2qk_2(1 - k_1)\cos(\vartheta + \alpha), \quad \ddot{X}_3 = 0$$

and system of Eq. (2.3) is written in the form

$$(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_1=0,\quad (\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_2=-mr\omega^2 qk_2(1-k_1)\cos(\vartheta+\alpha)$$
$$(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_3=0$$
$$l(\Delta\mathbf{N}_1-\Delta\mathbf{N}_2)\xi_3+r(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_2=A\ddot{\theta}-B\omega\dot{\psi}+mg(\frac{b}{\cos\beta}+l\operatorname{ctg}\beta-r)\theta$$
$$r(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_1=M_2$$
$$l(\Delta\mathbf{N}_1 -- \Delta\mathbf{N}_2)\xi_1=-A\ddot{\psi}-B\omega\dot{\theta}+mgl(1-k_1)\operatorname{tg}\beta\psi \tag{2.4}$$

Eqs (2.2) and (2.4) are used to determine the perturbations of the constraint reactions

$$(-1)^{k+1}\Delta\mathbf{N}_k\xi_1=-\frac{1}{2}q\sin(\vartheta+\alpha)\sqrt{\frac{k_2}{k_1}}[mg(1-k_1)\operatorname{tg}\beta+\frac{k_1\omega^2}{l}(Ak_2-B)],\quad M_2=0$$
$$(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_2=-mr\omega^2 qk_2(1-k_1)\cos(\vartheta+\alpha)$$
$$(-1)^{k+1}\Delta\mathbf{N}_k\xi_3=\frac{q}{2l}\cos(\vartheta+\alpha)[mr^2\omega^2k_2(1-k_1)-k_2\omega^2(Ak_1-B)+mg(\frac{b}{\cos\beta}+l\operatorname{ctg}\beta-r)]$$
$$k=1,2 \tag{2.5}$$

In the problem under consideration we can determine only the sum of the projections of the perturbations of the constraint reactions onto the OX_2 axis, since the rails and the wheelset are absolutely rigid bodies that come into contact at two points, and the problem of determining the reactions at each of them is ill-posed without additional assumptions.

3. The creep hypothesis taking into account the elasticity at the contact points and the stability of the rectilinear motion

According to Rocard's creep hypothesis, a wheel in contact with a rail has elastic compliance in the contact area, which manifests itself in strains in the vicinity of the contact points. In the case of absolutely rigid wheels rolling on absolutely rigid rails, the contact points move along the surfaces of the rails. When there are reactions that are orthogonal to the velocities of the contact points, small creep velocities appear, which are directed opposite to these reactions and are proportional to the magnitude of the components of the reactions and to the velocities of the contact points. The trajectories of the contact points on the rails are represented by formulae (1.1) and (1.3). According to the foregoing analysis, all the quantities appearing in these formulae depend on time and the two parameters q and α. Using Euler's formula (1.16), we find the projection of the velocity of the point C onto the OX_1 axis:

$$\dot{X}_1=r\omega+l\omega k_2\theta+l\dot{\psi}\Rightarrow X_1=r\omega t+\text{const}$$

We introduce the orthonormalized bases ($\boldsymbol{\tau}_k$, $\mathbf{n}_k$, $\mathbf{b}_k$) at the wheel/rail contact points, where

$$\boldsymbol{\tau}_k=\frac{\dot{\mathbf{R}}_{kr}}{|\dot{\mathbf{R}}_{kr}|}\cong\xi_1-\chi(\cos\gamma_k\xi_2+\sin\gamma_k\xi_3),\quad \chi=\frac{b\dot{\theta}}{r\omega}$$
$$\mathbf{n}_k=-\sin\gamma_k\xi_2+\cos\gamma_k\xi_3$$
$$\mathbf{b}_k=[\mathbf{n}_k\times\boldsymbol{\tau}_k]\cong\chi\xi_1+\cos\gamma_k\xi_2+\sin\gamma_k\xi_3;\quad k=1,2 \tag{3.1}$$

The creep hypothesis for each wheel is formulated in the form

$$\frac{\partial\mathbf{R}_{kr}}{\partial q}\dot{q}+\frac{\partial\mathbf{R}_{kr}}{\partial\alpha}\dot{\alpha}=-cr\omega(\mathbf{N}_k,\mathbf{b}_k)\mathbf{b}_k;\quad k=1,2 \tag{3.2}$$

where c is the compliance coefficient for lateral displacements of the wheel relative to the rail in the contact area and $r\omega=V$ is the absolute value of the velocity of the centre of mass of the wheelset in a projection onto the OX_1 axis. The system of vector equations (3.2) corresponds to the system of scalar equations

$$a_{11k}\dot{q}+a_{12k}\dot{\alpha}=-cr\omega(\mathbf{N}_k,\mathbf{b}_k),\quad a_{11k}=\left(\frac{\partial\mathbf{R}_{kr}}{\partial q},\mathbf{b}_k\right),\quad a_{12k}=\left(\frac{\partial\mathbf{R}_{kr}}{\partial\alpha},\mathbf{b}_k\right)$$
$$a_{21k}\dot{q}+a_{22k}\dot{\alpha}=0,\quad a_{21k}=\left(\frac{\partial\mathbf{R}_{kr}}{\partial q},\boldsymbol{\tau}_k\right),\quad a_{22k}=\left(\frac{\partial\mathbf{R}_{kr}}{\partial\alpha},\boldsymbol{\tau}_k\right);\quad k=1,2 \tag{3.3}$$

from which the derivatives $\dot{q}$ and $\dot{\alpha}$ can be determined:

$$\dot{q}=-\frac{cr\omega}{\Delta_k}a_{22k}(\mathbf{N}_k,\mathbf{b}_k),\quad \dot{\alpha}=\frac{cr\omega}{\Delta_k}a_{21k}(\mathbf{N}_k,\mathbf{b}_k),\quad \Delta_k=a_{11k}a_{22k}-a_{12k}a_{21k} \tag{3.4}$$

We will use formulae (1.1), (1.3), (1.7), (1.8), (1.14) and (3.1) to calculate the coefficients in Eq. (3.4), neglecting the small higher-order terms,

$$a_{11k} \cong -b\frac{\partial\theta}{\partial q}, \quad a_{12k} \cong -b\frac{\partial\theta}{\partial\alpha}, \quad a_{21k} \cong (-1)^k l(1-k_1)\frac{\partial\psi}{\partial q},$$
$$a_{22k} \cong (-1)^k l(1-k_1)\frac{\partial\psi}{\partial\alpha}$$
$$\Delta_k \cong (-1)^k blq(1-k_1)\sqrt{k_2/k_1}; \quad k=1,2$$

and we rewrite Eq. (3.4) in the form

$$\dot{q} = -cr\omega b^{-1}(\mathbf{N}_k, \mathbf{b}_k)\cos(\vartheta+\alpha), \dot{\alpha} = -cr\omega b^{-1}q^{-1}(\mathbf{N}_k, \mathbf{b}_k)\sin(\vartheta+\alpha); \quad k=1,2 \tag{3.5}$$

The right-hand sides of Eq. (3.5) contain unknown projections of perturbations of the reactions at the wheel/rail contact points in projections onto the OX_2 axis. Summing the right- and left-hand sides of Eq. (3.5), we obtain equations with known right-hand side

$$\dot{q} = -\frac{cr\omega}{2b}[\chi(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_1 + (\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_2\cos\beta +$$
$$+(\Delta\mathbf{N}_1-\Delta\mathbf{N}_2)\xi_3\sin\beta]\cos(\vartheta+\alpha)$$
$$\dot{\alpha} = -\frac{cr\omega}{2bq}[\chi(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_1 + (\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\xi_2\cos\beta +$$
$$+(\Delta\mathbf{N}_1-\Delta\mathbf{N}_2)\xi_3\sin\beta]\sin(\vartheta+\alpha) \tag{3.6}$$

Next, using formulae (2.5), we rewrite system of equations (3.6) in the form

$$\dot{q} = \frac{cr\omega}{2b}(D_1\omega^2 - D_2)q\cos^2(\vartheta+\alpha), \quad D_1 = mrk_2(1-k_1)^2\cos\beta + k_2 l^{-1}(Ak_1 - B)\sin\beta$$
$$\dot{\alpha} = \frac{cr\omega}{4b}(D_1\omega^2 - D_2)\sin 2(\vartheta+\alpha), \quad D_2 = mg[(1-k_1)\cos\beta + bl^{-1}\mathrm{tg}\beta] \tag{3.7}$$

Eq. (3.7) contain periodic functions of time of the form cos ϑ and sin ϑ and have the standard form for applying the method of averaging over the "fast time" ϑ, since the value of the elastic compliance determines the dimensionless small parameter of the system *cmg*. As a result, the averaged Eq. (3.7), which describe the evolution of the amplitude and phase of the perturbed motions of the wheelset, take the form

$$\dot{q} = \frac{cr\omega}{4b}(D_1\omega^2 - D_2)q, \quad \dot{\alpha} = 0 \tag{3.8}$$

The character of the solution of the first equation in system (3.8) depends on the sign of the expression $D(\omega) = D_1\omega^2 - D_2$: if $D(\omega) < 0$, the zero solution is asymptotically stable ($\lim_{t\to\infty} q(t) = \infty$); if $D(\omega) > 0$, the zero solution is unstable ($\lim_{t\to\infty} q(t) = \infty$). The coefficients D_1 and D_2 are positive for parameter values that correspond to the practical design of rails and wheelsets. This means that the sign of the function $D(\omega)$ changes from negative to positive at the value of the angular velocity of the rolling of the wheel $\omega^* = \sqrt{D_2/D_1}$. When the inequality $\omega < \omega^*$ holds, asymptotic stability of the rectilinear rolling of the wheelset occurs, and when $\omega > \omega^*$, instability occurs. This conclusion follows from an analysis of the solutions of Eq. (3.8), obtained by using the method of averaging and linearization. We will find an explicit expression for the critical angular velocity of the wheelset ω^*. We have

$$\omega^{*2} = \frac{mg[(1-k_1)\cos\beta + bl^{-1}\mathrm{tg}\beta]}{k_2[l^{-1}(Ak_1 - B)\sin\beta + mr(1-k_1)^2\cos\beta]} \tag{3.9}$$

The critical angular velocity is determined by the geometrical parameters of the design of the wheels and the rails, as well as by the radii of inertia of the wheelset ρ_1 and ρ_2 ($A = m\rho_1^2$, $B = m\rho_2^2$). The elastic compliance in the wheel/rail contact area does not affect the character of the stability of the rectilinear motion, but dictates the value of the derivative of the amplitude of the lateral oscillations of the centre of mass of the wheelset. The value of the elastic compliance, which is inversely proportional to the value of the stiffness in the wheel/rail contact area, can be obtained from an experiment or a numerical calculation of the contact problem of the theory of elasticity. According to the second equation of the averaged system, the phase of the oscillations of the contact points in the perturbed motion does not vary.

Remark. The results obtained hold for angles $\beta > 0$, since the wheel surfaces become cylindrical when $\beta = 0$, the angle $\theta = 0$, and the angle ψ is constant under any motion. In this case the centre of mass of the wheelset moves in a straight line with a constant velocity if the moment of the external forces about the centre of mass of the wheelset equals zero.

As an example, we will consider the numerical values of the parameters[2]

$$r = 0.475\,\mathrm{m}, \quad l = 0.74\,\mathrm{m}, \quad b = 0.3\,\mathrm{m}, \quad \beta = 0.109\,\mathrm{rad}, m = 1300\,\mathrm{kg}, \quad A = 825\,\mathrm{kg\,m^2}, \quad B = 75\,\mathrm{kg\,m^2}$$

and using Eq. (3.9), we find

$$k_1 = 0.0703, \quad k_2 = 1.044, \quad \omega^* = 4.73\,\mathrm{s^{-1}} \Rightarrow V^* = r\omega^* = 2.25\,\mathrm{m\,s^{-1}}$$

If we take a value of the vertical load acting on the wheelset that is nine times greater than its weight, the critical velocity will increase three-fold and will be equal to $6.75\,\mathrm{m\,s^{-1}}$.

4. The periodic motion of a flanged wheelset

In the design of a railway wheel, there is an element called a flange, which constrains the lateral displacements of the wheelset and thereby prevents it from derailing. Fig. 2 shows a schematic representation of the right-hand wheel, which comes in contact with the rail at two points: the main contact at the point K_1 produces the reaction $\mathbf{N}_1$, and the lateral contact between the flange and the rail at the point K_3 produces the reaction $\mathbf{N}_3$. If the angle of rotation of the wheelset $\psi = 0$, contact between the flange of the right-hand wheel and the rail occurs at the point K_3, where $\theta = \theta_0$, and contact between the flange of the left-hand wheel and the rail occurs at the point K_4, where $\theta = -\theta_0$. We will use $\mathbf{n}_3$ and $\mathbf{n}_4$ to denote the normals to the rails at the contact points K_3 and K_4, assuming that $\mathbf{n}_3 = (0, -\sin\kappa, \cos\kappa)$ and $\mathbf{n}_4 = (0, \sin\kappa, \cos\kappa)$ in the fixed system of coordinates $OX_1X_2X_3$ and that the coordinates of these points in the system of coordinates $Cx_1x_2x_3$, which is connected to the wheelset, are $(0, l_1, -r_1)$ and $(0, -l_1, -r_1)$, respectively. If contact between one of the flanges and a rail occurs at a certain time when the angle $\psi = 0$, this contact will subsequently cease to exist due to the difference between the rolling radii of the right- and left-hand wheels, and the wheelset motion will be described by the equations in Section 3. In the case when the angular velocity of the wheelset $\omega < \omega^*$, its rectilinear motion is stable. This means that the amplitudes of the oscillations of the angles θ and ψ will decrease, tending to zero, and there will subsequently be no flange/rail contact. In the opposite case, where ω is greater than the critical angular velocity ω^*, the amplitude of the oscillations of the angle θ will increase, and flange/rail contacts will appear for values of the variable $(\vartheta + \alpha)$ close to πk, $k \in \mathbf{Z}$.

Consider the periodic motion of a wheelset that occurs in this case. Suppose $\vartheta(0) = \alpha(0) = 0$, $q(0) = \theta_0$, and $\psi(0) = 0$ at the starting time. The contact between the flange of the left-hand wheel and the rail ceases to exist, and the parameters describing the wheelset motion vary according to Eq. (3.7), which can be represented in an approximation in the form (the prime denotes a derivative with respect to ϑ)

$$q' = P(\omega)\theta_0\cos^2\vartheta, \quad \alpha' = P(\omega)\sin\vartheta\cos\vartheta, \quad P(\omega) = \frac{cr}{2bk_1k_2}(D_1\omega^2 - D_2) \tag{4.1}$$

Eq. (4.1) were obtained by replacing the variables q and α with their initial conditions on the right-hand sides of Eq. (3.7), since the changes in these variables in a time equal to half the period π/ω are small. Note that Eq. (4.1) will be used at times less than half the period until the flange of the left-hand wheel comes into contact with the rail. The solution of system of Eq. (4.1) has the form

$$q = \theta_0\left[1 + \frac{1}{2}P(\omega)\left(\vartheta + \frac{1}{2}\sin 2\vartheta\right)\right], \quad \alpha = \frac{1}{2}P(\omega)\sin^2\vartheta$$

We will next find

$$\theta = q\cos(\vartheta + \alpha) = \theta_0\left[1 + \frac{1}{2}P(\omega)\left(\vartheta + \frac{1}{2}\sin 2\vartheta\right)\right]\cos\left[\vartheta + \frac{1}{2}P(\omega)\sin^2\vartheta\right]$$

$$\psi = -\sqrt{\frac{k_2}{k_1}}q\sin(\vartheta + \alpha) = -\theta_0\sqrt{\frac{k_2}{k_1}}\left[1 + \frac{1}{2}P(\omega)\left(\vartheta + \frac{1}{2}\sin 2\vartheta\right)\right]\sin\left[\vartheta + \frac{1}{2}P(\omega)\sin^2\vartheta\right]$$

Contact between the flange of the left-hand wheel and the rail occurs at the time $\vartheta_1 = \pi - \varepsilon$, when the condition $\theta = -\theta_0 \Rightarrow \varepsilon \cong \sqrt{\pi P(\omega)}$ holds.

At that time the variables ψ and α will be as follows:

$$\psi_0 = -\sqrt{\frac{k_2}{k_1}}q\sin(\vartheta + \alpha) \cong -\theta_0\sqrt{\frac{k_2}{k_1}}\varepsilon, \quad \alpha \cong \frac{1}{2}P(\omega)\varepsilon^2$$

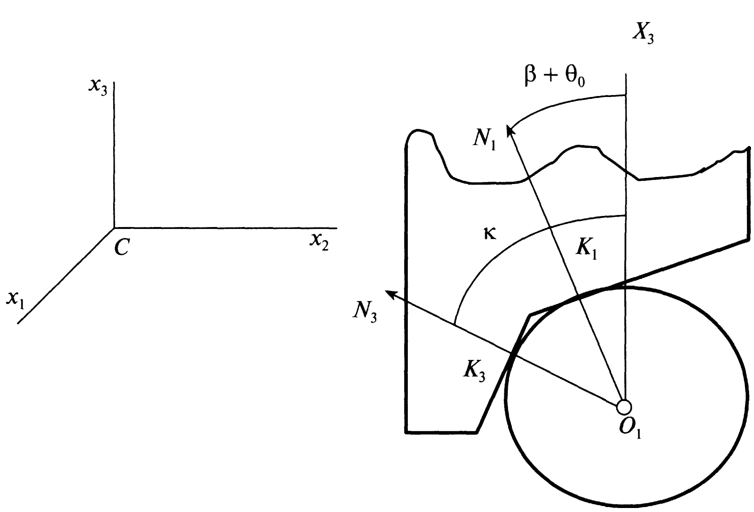

Fig. 2.

A further increase in the angle ψ will occur at the rate $\omega(r_2-r_1)/(2l)$, where r_1 and r_2 are the rolling radii of the right- and left-hand wheels, respectively, for lateral displacement of the wheelset. Using Eq. (1.9), we obtain

$$\frac{r_2-r_1}{2l}=\frac{(u_2-u_1)\sin\beta}{2l}=k_2\theta_0 \Rightarrow \psi=\psi_0+\theta_0\sqrt{k_2/k_1}(\vartheta+\varepsilon-\pi)$$

Contact between the flange of the left-hand wheel and the rail occurs in the dimensionless time range $[\vartheta_1, \vartheta_2]$, where $\vartheta_2=\pi-\varepsilon-\psi_0\theta_0^{-1}\sqrt{k_1/k_2}\cong\pi$. The angle $\theta=-\theta_0$ at this stage of rolling of the wheelset with contact between the left-hand flange and the rail. Motion of this kind can occur only with corresponding changes in q and α within the creep model when the wheels roll along rails. It is assumed that slipping, which produces tangential Coulomb dry friction forces, occurs at the contact point K_4.

When the angle ψ takes a zero value, the flange/rail contact ceases. The wheelset motion, taking the creep phenomenon into account, will be described by Eq. (4.1) with the initial conditions $q(\pi)=\theta_0$ and $\alpha(\pi)=0$. An increase in the amplitude $q(t)$ causes the angle θ to become equal to θ_0 at a time close to 2π. At this time contact between the flange of the right-hand wheel occurs at the point K_3. This is followed by a stage of motion with slipping at the contact point K_3, which will continue until the angle ψ vanishes. This marks the end of a period of motion of the wheelset that contains two intervals in which flange/rail contact occurs and two intervals in which there is no contact.

We will formulate the creep hypothesis in the time interval $[\vartheta_1, \vartheta_2]$ taking into account the reaction at the contact point K_4

$$\mathbf{N}_4=N_4(\mathbf{n}_4+f\boldsymbol{\xi}_1),\quad \mathbf{n}_4=(0,\sin\kappa,\cos\kappa)$$

where f is the coefficient of dry friction, since slipping is assumed to occur at the contact point K_4. We obtain the perturbations of the reactions by analogy with Eq. (2.3), taking into account the laws of variation of the coordinates X_1, X_2 and X_3 and the angles θ and ψ at the stage of slipping of the flange along the rail

$$\begin{aligned}
&(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\boldsymbol{\xi}_1+fN_4=0\\
&(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\boldsymbol{\xi}_2+N_4\sin\kappa=mr\theta_0\omega^2k_2(1-k_1)\\
&(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\boldsymbol{\xi}_3+N_4\cos\kappa=0\\
&A\theta_0\omega^2k_1k_2-B\omega^2\theta_0k_2-mg(l\,\mathrm{ctg}\beta+b\cos^{-1}\beta-r)\theta_0=\\
&=l(\Delta\mathbf{N}_1-\Delta\mathbf{N}_2)\boldsymbol{\xi}_3+r(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\boldsymbol{\xi}_2+N_4(-l_1\cos\kappa+r_1\sin\kappa)\\
&M_2-r(\Delta\mathbf{N}_1+\Delta\mathbf{N}_2)\boldsymbol{\xi}_1-fN_4r_1=0\\
&mgl\,\mathrm{tg}\beta(1-k_1)\theta_0\sqrt{k_2/k_1}(\vartheta-\pi)-l(\Delta\mathbf{N}_1-\Delta\mathbf{N}_2)\boldsymbol{\xi}_1+fN_4l_1=0
\end{aligned}\tag{4.2}$$

In the last equation in system (4.2), we omitted the terms $A\ddot{\psi}+B\omega\dot{\theta}$, which are second-order terms at the stage of rolling of the wheelset with the flange/rail contact. Eq. (4.2) were written in the assumption that the angular velocity $\dot{\varphi}=\omega$ is constant, which can be achieved by appropriately selecting the moment of the external forces M_2. As before, the unperturbed motion is described by the relations in Section 2. As a result, according to what was stated in Section 3, we obtain equations of the type (3.6), in which the perturbations of the constraint reactions $\Delta\mathbf{N}_1$ and $\Delta\mathbf{N}_2$ are determined from Eq. (4.2), which contain the unknown quantity N_4. In addition, we should set $\chi=0$ in Eq. (3.6). The system of equations (3.6) and (4.2) becomes closed if we take into account the equation $q\cos(\vartheta+\alpha)=\theta_0$, which expresses the constancy of the angle θ at the stage when there is the flange/rail contact.

We express the variations in the constraint reactions at the contact points K_1 and K_2 using Eq. (4.2), substitute the expressions for them into Eq. (3.6) and find

$$\begin{aligned}
&\dot{q}=\frac{cr\omega}{2b}H\cos(\vartheta+\alpha),\quad \dot{\alpha}=\frac{cr\omega}{2bq}H\sin(\vartheta+\alpha),\quad q\cos(\vartheta+\alpha)=\theta_0\\
&H=\theta_0(\omega^2D_1-D_2)-D_3N_4,\\
&D_3=(1-k_1)\sin\kappa\cos\beta-l_1l^{-1}\cos\kappa\sin\beta+r_1l^{-1}\sin\kappa\sin\beta
\end{aligned}\tag{4.3}$$

We differentiate the third equation in (4.3) with respect to time, replace the derivatives in the equality obtained $\dot{q}\cos(\vartheta+\alpha)-q\sin(\vartheta+\alpha)(\dot{\vartheta}+\dot{\alpha})=0$ using the first two equalities and write the result in the form

$$\frac{cr}{2b}H\cos2(\vartheta+\alpha)-\theta_0\sqrt{k_1k_2}\,\mathrm{tg}(\vartheta+\alpha)=0\tag{4.4}$$

In the time interval $[\vartheta_1, \vartheta_2]$, in which the flange/rail contact occurs, the angle $\vartheta+\alpha$ differs from π by a small quantity of the order of ε. Taking this into account and setting $\cos2(\vartheta+\alpha)\approx1$ and $\mathrm{tg}(\vartheta+\alpha)\approx\vartheta-\pi$, from Eq. (4.4) we obtain the following relation, up to small second-order terms:

$$N_4=\frac{\theta_0}{D_3}\left[D_1\omega^2-D_2+\frac{2b}{cr}\sqrt{k_1k_2}(\pi-\vartheta)\right],\quad \vartheta_1<\vartheta<\pi\tag{4.5}$$

The reaction N_4 must be greater than zero. This condition holds if the rolling angular velocity of the wheelset ω is greater than the critical angular velocity ω^*, which corresponds to loss of stability of the rectilinear rolling of the wheelset. The following inequality holds

$$\frac{\theta_0}{D_3}(D_1\omega^2-D_2)\le N_4\le\frac{\theta_0}{D_3}\left[D_1\omega^2-D_2+\sqrt{\frac{2\pi b}{cr}(D_1\omega^2-D_2)}\right]$$

The reaction takes a maximum value at the time of the flange/rail contact and then decreases linearly until contact ceases. It follows from formula (4.5) that the reaction at the flange/rail contact point and the duration of this contact increase as the rolling angular velocity of the wheel increases.

The force acting on the rail at the contact point K_4 can cause lateral displacement of the rail at a fairly large value of the angular velocity ω. This circumstance imposes a constraint on the rolling velocity of the wheelset. Let N_0 be the maximum attainable value of the normal magnitude of the lateral reaction. Then the maximum possible rolling angular velocity of the wheelset can be calculated from the formula

$$\omega_{\max} = \left(\frac{D_2}{D_1} + \frac{\pi b}{2crD_1}\left[\sqrt{1 + \frac{2N_0 D_3 cr}{\pi b \theta_0}} - 1\right]^2\right)^{1/2}$$

5. The rolling resistance of the wheelset

The rolling resistance of the wheelset is affected by two factors: the possible loss of velocity when a flange of the wheelset strikes the rail at the time of contact (the impact is assumed to be absolutely inelastic) and the action of the dry friction forces at the contact point K_4 at the stage of rolling of the flange with slipping relative to the rail. We will examine these two phenomena and find the resistance force.

At the time of contact of the flange of the right-hand wheel with the rail, constraints, under which the velocity of the centre of mass of the wheelset and its angular velocity change abruptly, are imposed. Impact effects on the system are possible only at the three wheelset/rail contact points, since the remaining external forces are limited. Using system of Eq. (2.3), we write the impact theory equations, assuming that the impact is absolutely inelastic[6]

$$\begin{aligned} &m\Delta\dot{X}_1 = 0, \quad m\Delta\dot{X}_2 = -P_1\sin(\beta - \theta_0) + P_2\sin(\beta + \theta_0) + P_4\sin\kappa \\ &m\Delta\dot{X}_3 = P_1\cos(\beta - \theta_0) + P_2\cos(\beta + \theta_0) + P_4\cos\kappa \\ &A\Delta\dot{\theta} = l[P_1\cos(\beta - \theta_0) - P_2\cos(\beta + \theta_0)] - r[P_1\sin(\beta - \theta_0) - P_2\sin(\beta + \theta_0)] + \\ &+ P_4(-l_1\cos\kappa + r_1\sin\kappa), \quad B\Delta\dot{\varphi} = 0, \quad A\Delta\dot{\psi} = 0 \end{aligned} \tag{5.1}$$

Here $P_i\mathbf{n}_i$ (i = 1, 2, 4) are the normal impact impulses of the constraints. Equations (5.1) do not contain impact impulses belonging to the planes tangential to the rails at the contact points K_1, K_2 and K_4 that are generated by dry friction forces, because it is assumed that these forces, remain finite.

We will find the velocity jumps that are still unknown in system of Eq. (5.1). Using relations (1.17) and discarding second-order terms, we obtain

$$\Delta\dot{\theta} = 0 - \theta_0\omega\sqrt{k_1k_2}\sin(\vartheta_1 - \pi) \approx 0, \quad \Delta\dot{X}_2 = -r\Delta\dot{\theta} \approx 0, \quad \Delta\dot{X}_3 = -l\Delta\dot{\theta} \approx 0$$

A system of homogeneous linear equations in the unknown values of the impact impulses P_1, P_2 and P_3 can be isolated from system of Eq. (5.1). This system has a zero solution, since its determinant is non-zero. Thus, we conclude that the initial times of the flange/rail contact are not accompanied by impact phenomena within the approximations considered.

We will determine the value of the moment of the external forces M_2 needed to maintain a constant velocity of the centre of mass of the wheelset. Taking into account the expression for the magnitude of the reaction (4.5), from the first and fifth equations in system (4.2) we find

$$\begin{aligned} &M_2 = f(r_1 - r)N_4 = \Phi_1 + \Phi_2(\pi - \vartheta) \Rightarrow Q = \frac{1}{\sqrt{k_1k_2}}\int_{\vartheta_1}^{\pi} M_2 d\vartheta \\ &\Phi_1 = \frac{f\theta_0(r_1 - r)}{D_3}(D_1\omega^2 - D_2), \quad \Phi_2 = \frac{2bf\theta_0(r_1 - r)}{crD_3}\sqrt{k_1k_2} \end{aligned} \tag{5.2}$$

where Q is the work of the dry friction force during the flange/rail contact at the point K_4, which can be represented in the form

$$Q = \frac{1}{\sqrt{k_1k_2}}\left(\Phi_1\varepsilon + \frac{1}{2}\Phi_2\varepsilon^2\right) = \frac{\pi f\theta_0(r_1 - r)}{2k_1k_2D_3}(D_1\omega^2 - D_2)\left[\sqrt{\frac{2cr}{\pi b}(D_1\omega^2 - D_2) + 1}\right]$$

We will consider a numerical example, supplementing the data presented in Section 3. Suppose

$$\omega = 10\,\mathrm{s}^{-1}, \quad l_1 = 0.73\,\mathrm{m}, \quad r_1 = 0.485\,\mathrm{m}, \quad \kappa = \pi/4, \quad c = 10^{-7}\,N^{-1}$$

As a result we find

$$\begin{aligned} &\theta_0 = \frac{X_2 \mathrm{tg}\beta}{l(1 - k_1)} = 0.00248, \quad D_1 = 554\ \mathrm{Ns}^2, \quad D_2 = 12342\ \mathrm{N} \\ &P(10) = 0.04647, \quad \varepsilon = 0.382, \quad D_3 = 0.6281 \end{aligned}$$

Here $X_2 = l_1 - l = 0.02$ m. We next obtain the magnitude of the normal reaction at the contact point K_4 and its minimum and maximum values

$$N_4 = [170.1 + 1.35\cdot 10^4(\pi - \vartheta)]\mathrm{N}, \quad \min N_4 = 170\,\mathrm{N}, \quad \max N_4 = 5332\,\mathrm{N}$$

If we take the coefficient of dry friction $f = 0.15$, then, using formula (5.2), we find the minimum and maximum values of the moment of the external forces and the work of the dry friction forces

$$\min M_2 = f(r_1 - r)\min N_4 = 0.255\,\text{Nm}, \quad \max M_2 = f(r_1 - r)\max N_4 = 8\,\text{Nm}$$

$$Q = \frac{f(r_1 - r)(\max N_4 - \min N_4)\varepsilon}{2\sqrt{k_1 k_2}} = 5.46\,\text{Nm}$$

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